Subtraction and division operations over hesitant fuzzy sets

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Abstract. Hesitant fuzzy set (HFS), which permits the membership having a set of possible values, has turned out to be a powerful structure in expressing uncertainty and vagueness. In this paper, we propose two new basic operations over HFSs, which are the subtraction operation and the division operation. Several operational laws of these two operations over HFSs are given. The relationship between intuitionistic fuzzy set (IFS) and HFS is further verified in terms of these two operations. In addition, the relationships between these two operations are also established in this paper. The operations can be immediately extended into interval-valued hesitant fuzzy sets and dual hesitant fuzzy sets. The subtraction and division operations are significantly important in forming the integral theoretical framework of HFS and may have many practical applications in decision making.

Keywords: Hesitant fuzzy set, subtraction operation, division operation

1. Introduction

Hesitant fuzzy set (HFS) \cite{15}, as a new extension of classical fuzzy set (FS) \cite{25}, has attracted more and more scholars’ attention \cite{7\textsuperscript{a}–12, 14, 17–19, 23, 24, 26, 27}. The motivation for introducing such a set is that it is sometimes difficult to establish the membership degree of an element to a set, and in some circumstances this difficulty is caused by a doubt of several possible values. Torra \cite{15} firstly gave the concept of HFS, and defined some of its basic operations. Furthermore, Torra and Narukawa \cite{16} presented an extension principle permitting to generalize the existing operations on FSs to HFSs, and described the application of this new type of set in the framework of decision making. Xu and Xia \cite{23, 24} investigated the distance, similarity, and correlation measures for HFSs. In order to select the most desirable alternatives for multiple criteria decision making (MCDM) problems with hesitant fuzzy information, different types of aggregation operators have been proposed to fuse the multi-dimensional hesitant fuzzy values into overall values \cite{11, 17, 19, 26}. Since the preference relation is one of the most important and powerful tool in MCDM \cite{20, 22}, Xia and Xu \cite{18} proposed the concept of hesitant fuzzy preference relation and hesitant multiplicative preference relation. Liao et al. \cite{12} investigated the multiplicative consistency of a hesitant fuzzy preference relation and the group consensus among different decision makers. Chen et al. \cite{7} introduced the interval-valued hesitant preference relations and investigated their application in decision making. Many different decision making methods have also been proposed by the scholars. Chen et al. \cite{8} proposed the hesitant fuzzy ELECTRE method for MCDM. Liao and Xu \cite{9} extended the classical VIKOR method \cite{13} to accommodate hesitant fuzzy circumstances. In order to make a more reasonable decision, Liao and Xu \cite{10} introduced an interactive method based on some optimization models for MCDM problems with hesitant fuzzy information. Recently, HFS has been extended
into some different forms, such as the hesitant fuzzy linguistic term set (HFLTS) [14], dual hesitant fuzzy set (DHFS) [27], interval-valued hesitant fuzzy set (IVHFS) [7]. All of these achievements have shown that HFS has turned out to be a powerful technique in representing uncertainty and vagueness.

In Torra [15]'s original paper, he defined the complement, union and intersection of HFS, and also established the relationship between HFS and intuitionistic fuzzy set (IFS) [1, 3], based on which, Xia and Xu [17] gave some operational laws over HFSs, such as addition and multiplication. However, even till now, as far as we know, there is no any investigation on the subtraction and division operations over HFSs. The subtraction and division operations are significantly important in forming the integral theoretical framework of HFSs. Meanwhile, it is also critical in developing some well-known decision making method such as PROMETHEE [5] with hesitant fuzzy information. Hence, in this paper, we shall define these operations over HFSs. The motivation for introducing the subtraction and division operations for HFS is also based on the relationship between HFS and IFS. HFS encompasses IFS as a particular case and the envelope of a HFS is an IFS [15]. The subtraction and division operations of IFS has been investigated by some scholars [2, 4, 6] and finally established by Atanassov [3] in his recent published book. Since these two operations are also important for HFS, we shall try to develop the two new operations for HFS.

The remainder of this paper is set out as follows: Section 2 gives some basic knowledge on HFS and their basic operations. Section 3 proposes the subtraction operation and division operation over HFSs. Several operational laws of these two operations are given. The relationships between IFSs and HFSs are further verified in terms of these two operations. In addition, the relationships between these two operations are also established in this paper. The paper ends with some conclusions in Section 4.

2. Preliminaries

We recall some relevant basic preliminaries in this section.

2.1. Hesitant fuzzy set

Intuitionistic fuzzy set (IFS), which assigns to each element a membership degree, a non-membership degree and a hesitancy degree, is more powerful in dealing with vagueness and uncertainty than FS.

Definition 1. [1, 3] Let a crisp set $X$ be fixed and let $A \subseteq X$ be a fixed set. An intuitionistic fuzzy set (IFS) $\tilde{A}$ in $X$ is an object of the following form:

$$\tilde{A} = \{(x, \mu_A(x), \nu_A(x)) | x \in X\} \quad (1)$$

where the function $\mu_A : E \rightarrow [0, 1]$ and $\nu_A : E \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set $A$, respectively, and for every $x \in X$, $0 \leq \mu_A + \nu_A \leq 1$ holds. For each IFS $\tilde{A}$ on $X$, then

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \quad (2)$$

is called the degree of non-determinacy (uncertainty) of the membership of the element $x \in X$ to the set $A$. In the case of ordinary fuzzy sets, $\pi_A(x) = 0$ for every $x \in X$.

However, when giving the membership degree of an element, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. In such cases, hesitant fuzzy set (HFS), as new generalization of IFS, which permits the membership of an element of a given set represented by several possible values between 0 and 1, is capable to determine the membership degree especially when we have several different values on it.

Definition 2. [15] Let $X$ be a fixed set, a HFS on $X$ is in terms of a function $h$ that when applied to $X$ returns a subset of $[0, 1]$, which can be represented in terms of the following mathematical symbol:

$$E = \{x, h_E(x) > | x \in X\} \quad (3)$$

where $h_E(x)$ is a set of values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set $E$.

For convenience, Xia and Xu [17] called $h_E(x)$ a hesitant fuzzy element (HFE), which denotes a basic component of the HFS. HFS encompasses IFS as a particular case, and it is a particular case of type 2 fuzzy set. The typical HFS is the one where $h_E(x)$ is finite. Torra (2010) gave some special HFEs for $x$ in $X$:

1. Empty set: $h(x) = \emptyset$, we denotes it as $O^*$ for simplification;
2. Full set: $h(x) = \{1\}$, denoted as $E^*$;
Given an intuitionistic fuzzy value (IFV; [21]) 

\( h(x) = \langle \mu_A(x), \nu_A(x) \rangle \), its corresponding HFE is straightforward: 

\[
\nuA = \{ \mu_A(x), 1 - \nu_A(x) \}, \text{ if } \mu_A(x) \neq 1 - \nu_A(x). \]

But, the construction of IFV from HFE is not so easy when the HFE contains more than one number for each \( x \). As for this issue, Torra [15] showed that the envelope of an HFE is an IFV, expressed in the following definition:

**Definition 4.** [15] For three HFEs \( h, h_1, h_2 \), the following operations are defined:

1. \( h^- = \{ \emptyset \} \), \( h^+ = \{ \emptyset \} \), with \( h^- = \min \{ \gamma | \gamma \in h \} \) and \( h^+ = \max \{ \gamma | \gamma \in h \} \).

2. Basic operations for HFSs:

   Torra [15] defined some operations such as complement, union and intersection about HFEs.

   **Definition 5.** [15] Given a HFE \( h \), we define the IFV \( A_{env}(h) \) as the envelope of \( h \), where \( A_{env}(h) \) can be represented as \( (h^-, h^+) \), with \( h^- = \min \{ \gamma | \gamma \in h \} \) and \( h^+ = \max \{ \gamma | \gamma \in h \} \).

   \( A_{env}(h) \) is an IFV, expressed in the following definition:

   **Definition 2.** [15] Let \( h, h_1 \) and \( h_2 \) be two HFEs with \( h(x) \) a nonempty convex set for all \( x \) in \( X \), \( h_1 \) and \( h_2 \) are IFVs. Then,

   \[
   \text{(1) } h^- = \{ \emptyset \} , \quad \text{(2) } h^+ = \{ \emptyset \} .
   \]

   \[
   \text{(3) Proposition 1.} \quad \text{Let } h, h_1 \text{ and } h_2 \text{ be three HFEs. Then,}
   \]

   \[
   \text{(1) } A_{env}(h) = (A_{env}(h))^c ,
   \]

   \[
   \text{(2) } A_{env}(h_1 \cup h_2) = A_{env}(h_1) \cup A_{env}(h_2) ,
   \]

   \[
   \text{(3) } A_{env}(h_1 \cap h_2) = A_{env}(h_1) \cap A_{env}(h_2) .
   \]

   **Proposition 2.** [15] Let \( h_1 \) and \( h_2 \) be two HFEs with \( h(x) \) a nonempty convex set for all \( x \) in \( X \), i.e., \( h_1 \) and \( h_2 \) are IFVs. Then,

   \[
   \text{(1) } h_1 \cap h_2 \text{ is equivalent to HFS complement;}
   \]

   \[
   \text{(2) } h_1 \cup h_2 \text{ is equivalent to HFS intersection;}
   \]

   \[
   \text{(3) } h_1 \cup h_2 \text{ is equivalent to HFS union.}
   \]

   The above proposition reveals the operations defined for HFSs are consistent with the ones for IFSs. Based on the relationships between HFSs and IFSs, Xia and Xu [17] gave some operational laws on the HFSs \( h_1, h_2 \).

   **Definition 5.** [17] Let \( h, h_1 \) and \( h_2 \) be three HFEs, and \( \lambda \) be a positive real number, then

   \[
   \text{(1) } h^\lambda = \cup \{ \emptyset \} ;
   \]

   \[
   \text{(2) } h \lambda = \cup \{ 1 - (1 - \gamma)^\lambda \} ;
   \]

   \[
   \text{(3) } h_1 \oplus h_2 = \cup \{ \gamma_1 + \gamma_2 - \gamma_1 \gamma_2 \} ;
   \]

   \[
   \text{(4) } h_1 \ominus h_2 = \cup \{ \gamma_1 (1 - \gamma_2) \} .
   \]

   However, neither Torra [15] nor Xia and Xu [17] pay any attention on the subtraction and division operations over HFSs. The subtraction and division operations are significantly important in forming the integral theoretical framework of HFSs. Meanwhile, it is also an indispensable foundation in developing some well-known decision making method such as PROMETHEE with hesitant fuzzy information. Hence, in the following of this paper, we shall define these basic operations over HFSs.

3. Subtraction and division operations over HFSs

Considering the relationships between IFSs and HFSs, to start our investigation, let’s first review the subtraction and division operations over IFSs. The subtraction and division operations were firstly proposed by Atanassov and Riečan [4]. Later, Chen [6] also introduced these operations for IFSs, which were derived from the deconvolution for equations using addition and multiplication operations of IFSs, and the forms of these two operations they proposed were similar to those of Atanassov and Riečan [4]. Based on the different versions of operation “negation”, Atanassov [2] further developed a family of different kinds of subtraction operations for IFSs. Among all these different subtraction operations, Atanassov [3] finally chose the following forms as the standard definition for subtraction and division operations over IFSs in his recent published book.

**Definition 6.** [3] For every two given IFSs \( A \) and \( B \), the subtraction and division operations have the forms:

\[
A \ominus B = \{ \{ x, \ \mu_{A \ominus B}(x), \ \nu_{A \ominus B}(x) \} | x \in E \} .
\]
where

\[
\mu_{A \odot B}(x) = \begin{cases} 
\frac{\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \text{ and } \nu_A(x) > 0 \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
\nu_{A \odot B}(x) = \begin{cases} 
\frac{\nu_A(x) \cdot \nu_B(x)}{\nu_A(x) + \nu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \text{ and } \nu_A(x) > 0 \\
1, & \text{otherwise}
\end{cases}
\]

and

\[
A \varnothing B = \{(x, \mu_{A \odot B}(x), \nu_{A \odot B}(x)) | x \in E\}
\]

where

\[
\mu_{A \odot B}(x) = \begin{cases} 
\frac{\mu_A(x) \cdot \nu_B(x)}{\mu_A(x) + \nu_B(x)}, & \text{if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) > 0 \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
\nu_{A \odot B}(x) = \begin{cases} 
\frac{\nu_A(x) \cdot \mu_B(x)}{\nu_A(x) + \mu_B(x)}, & \text{if } \mu_A(x) \leq \nu_B(x) \text{ and } \mu_A(x) > 0 \\
1, & \text{otherwise}
\end{cases}
\]

Inspired by Definition 6 and based on the relationships between IFSs and HFSs, we can introduce the definition of subtraction and division operations over HFSs.

**Definition 7.** Let \( h, h_1 \) and \( h_2 \) be three HFEs, then we introduce the following two operations over HFSs:

1. \( h_1 \triangleleft h_2 = \cup_{\nu_B \geq \nu_A} \{t\} \), where
   \[
t = \begin{cases} 
   \frac{\nu_B - \nu_A}{\nu_2 - \nu_1}, & \text{if } \nu_1 \geq \nu_2 \text{ and } \nu_2 \neq 1 \\
   1, & \text{otherwise}
   \end{cases}
   \]

2. \( h_1 \varnothing h_2 = \cup_{\nu_B \geq \nu_A} \{t\} \), where
   \[
t = \begin{cases} 
   \frac{\nu_B - \nu_A}{\nu_2 - \nu_1}, & \text{if } \nu_1 \geq \nu_2 \text{ and } \nu_2 \neq 0 \\
   1, & \text{otherwise}
   \end{cases}
   \]

To make it more adequate, we let \( h \odot U^* = O^*; h \odot U^* = O^* \). According to this definition, it is obvious that for any HFE \( h \), the following equations hold:

- \( h \odot h = O^* \);
- \( h \odot h = O^*; h \odot E^* = O^* \);
- \( h \odot E^* = O^*; h \odot O^* = O^* \);
- \( h \odot O^* = O^*; h \odot E^* = O^* \).

In addition, it follows from above that some special cases hold:

- \( E^* \odot E^* = O^*; U^* \odot E^* = O^*; O^* \odot E^* = O^* \);
- \( E^* \odot U^* = O^*; U^* \odot U^* = O^* \);
- \( E^* \odot E^* = O^*; U^* \odot U^* = O^* \).

For the brevity of presentation, in the process of theoretical derivation thereafter, we shall not consider the particular case where \( t = 0 \) in subtraction operation and \( t = 1 \) in division operation. It is noted that HFS encompasses IFS as a particular case; thus the subtraction and division operations over HFSs should be equivalent to the subtraction and division operations over IFSs when not considering the non-membership degree of each IFS. Comparing Definition 6 and Definition 7, we can see that this requirement is met. The following theorems show that our proposed subtraction and division operations over HFSs in Definition 7 are convincing and they satisfy some basic properties.

**Theorem 1.** Let \( h_1 \) and \( h_2 \) be two HFEs, then

1. \( (h_1 \odot h_2) \odot h_2 = h_1 \), if \( \nu_1 \geq \nu_2; \nu_2 \neq 1 \);
2. \( (h_1 \odot h_2) \odot h_2 = h_1 \), if \( \nu_1 \geq \nu_2; \nu_2 \neq 0 \).

**Proof.** For two HFEs \( h_1 \) and \( h_2 \), we have

\[
(1) (h_1 \odot h_2) \odot h_2 = \cup_{\nu_B \geq \nu_A} \{t\} \neq 1 \left\{ \begin{array}{l}
\frac{\nu_B - \nu_A}{\nu_2 - \nu_1} \\
1 - \nu_2
\end{array} \right\} \odot h_2
\]

\[
= \cup_{\nu_B \geq \nu_A} \{t\} \neq 1 \left\{ \begin{array}{l}
\frac{\nu_B - \nu_A}{\nu_2 - \nu_1} \\
1 - \nu_2
\end{array} \right\}
\]

\[
= \left\{ \begin{array}{l}
\frac{\nu_B - \nu_A}{\nu_2 - \nu_1} \\
1 - \nu_2
\end{array} \right\}
\]

\[
= \left\{ \begin{array}{l}
\frac{\nu_B - \nu_A}{\nu_2 - \nu_1} \\
1 - \nu_2
\end{array} \right\}
\]

\[
= \left\{ \begin{array}{l}
\frac{\nu_B - \nu_A}{\nu_2 - \nu_1} \\
1 - \nu_2
\end{array} \right\}
\]
Theorem 2. Let $h_1$ and $h_2$ be two HFEs, $\lambda > 0$, then

1. $\lambda (h_1 \odot h_2) = \lambda h_1 \odot \lambda h_2$, if $y_1 \geq y_2$, $y_2 \neq 1$;
2. $(h_1 \odot h_2)^\lambda = h_1^\lambda \odot h_2^\lambda$, if $y_1 \geq y_2$, $y_2 \neq 0$.

Proof. For two HFEs $h_1$ and $h_2$, we have

1. $\lambda (h_1 \odot h_2) = \lambda \cdot \cup_{h \in h_1 \cap h_2} \gamma_1 \geq 0, y_2 \neq 0 \neq 1 \left\{ \frac{y_1 - y_2}{y_2} \right\}$
2. $(h_1 \odot h_2)^\lambda = \cup_{h \in h_1 \cap h_2} \gamma_1 \geq 0, y_2 \neq 0 \neq 1 \left\{ \frac{y_1 - y_2}{y_2} \right\}^\lambda$.

Since $y = x^\lambda (\lambda > 0)$ is a monotonic increasing function when $x > 0$, and also since $y_1 \geq y_2$, $y_2 \neq 1$, it follows that $1 - (1 - y_1)^\lambda > 1 - (1 - y_2)^\lambda$, $1 - (1 - y_2)^\lambda > 1$. Thus,

$\lambda (h_1 \odot h_2) = \cup_{h \in h_1 \cap h_2} \gamma_1 \geq 0, y_2 \neq 0 \neq 1 \left\{ \frac{1 - (1 - y_1)^\lambda}{1 - (1 - y_2)^\lambda} \right\}$

$= \cup_{h \in h_1 \cap h_2} \gamma_1 \geq 0, y_2 \neq 0 \neq 1 \left\{ \frac{(1 - y_1)^\lambda - (1 - y_2)^\lambda}{(1 - y_2)^\lambda} \right\} = \lambda (h_1 \odot h_2)$.

This completes the proof of the theorem.

Theorem 3. Let $h = \cup_{y \in h} [y]$ be a HFE, and $\lambda_1 \geq \lambda_2 > 0$ be real numbers, then

1. $\lambda_1 h \odot \lambda_2 h = (\lambda_1 - \lambda_2) h$, if $y \neq 1$;
2. $h^{\lambda_1} \odot h^{\lambda_2} = h^{\lambda_1 - \lambda_2}$, if $y \neq 0$.

Proof. For a HFE $h$ and $\lambda_1, \lambda_2 > 0$, we have

1. $\lambda_1 h \odot \lambda_2 h = \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_1} \right\} \odot \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_2} \right\}$
2. $h^{\lambda_1} \odot h^{\lambda_2} = \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_1} \right\} \odot \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_2} \right\}$.

Since $y = a^x (0 < a < 1)$ is a monotonic decreasing function when $x > 0$, and also since $\lambda_1 \geq \lambda_2$, $y \neq 1$, it follows that $1 - (1 - y)^{\lambda_1} \geq 1 - (1 - y)^{\lambda_2}$, $1 - (1 - y)^{\lambda_2} \neq 1$. Thus,

$\lambda_1 h \odot \lambda_2 h = \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_1} \right\} \odot \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_2} \right\}$

$= \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_1} \right\} \odot \cup_{y \in h} \left\{ 1 - (1 - y)^{\lambda_2} \right\}$

$= \lambda_1 h \odot \lambda_2 h$.

This completes the proof of the theorem.
Theorem 4. For three HFEs $h_1$, $h_2$, and $h_3$, the followings are valid:

1. $h_1 \circ h_2 \circ h_1 = h_1 \circ h_1 \circ h_2$, if $\gamma_1 \geq \gamma_2, \gamma_1 \geq \gamma_3, \gamma_1 \neq 1, \gamma_1 \neq 2, \gamma_1 - \gamma_2 + \gamma_3 \gamma_3 \geq 0$.
2. $h_1 \circ h_2 \circ h_3 = h_2 \circ h_3 \circ h_2$. If $\gamma_1 \leq \gamma_2 \gamma_3, \gamma_2 \neq 0, \gamma_3 \neq 0$.

Proof. For three HFEs $h_1$, $h_2$, and $h_3$, we have

1. Since $\gamma_1 - \gamma_2 - \gamma_3 + \gamma_3 \gamma_3 \geq 0$, we have $\gamma_1 - \gamma_2 - \gamma_3 + \gamma_3 \gamma_3 / 1 - \gamma_3 \geq 0$. Thus,

$$h_1 \circ h_2 \circ h_3 = \bigcup_{\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}, \gamma_1 \neq 1, \gamma_2 \neq 2, \gamma_3 \neq 0} \left\{ \frac{\gamma_1 - \gamma_2 - \gamma_3 + \gamma_3 \gamma_3}{1 - \gamma_3} \right\} \circ h_3$$

2. Since $\gamma_1 - \gamma_2 - \gamma_3 + \gamma_3 \gamma_3 \geq 0$, we also have $\gamma_1 - \gamma_2 - \gamma_3 + \gamma_3 \gamma_3 / 1 - \gamma_3 \geq 0$. Thus,

$$h_1 \circ h_1 \circ h_3 = \bigcup_{\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}, \gamma_1 \neq 1, \gamma_2 \neq 2, \gamma_3 \neq 0} \left\{ \frac{\gamma_1 - \gamma_2 - \gamma_3 + \gamma_3 \gamma_3}{1 - \gamma_3} \right\} \circ h_3$$

Thus, $h_1 \circ h_2 \circ h_1 = h_1 \circ h_1 \circ h_2$. If $\gamma_1 \leq \gamma_2 \gamma_3, \gamma_2 \neq 0, \gamma_3 \neq 0$.

Theorem 5. For three HFEs $h_1$, $h_2$, and $h_3$, the followings are valid:

1. $h_1 \circ h_2 \circ h_3 = h_1 \circ (h_2 \circ h_3)$, if $\gamma_1 \geq \gamma_2, \gamma_1 \geq \gamma_3, \gamma_1 \neq 1, \gamma_1 \neq 1, \gamma_1 - \gamma_2 + \gamma_2 \gamma_3 \gamma_3 \geq 0$.
2. $h_1 \circ h_2 \circ h_3 = h_1 \circ (h_2 \circ h_3)$. If $\gamma_1 \leq \gamma_2 \gamma_3, \gamma_2 \neq 0, \gamma_3 \neq 0$.

Proof. For three HFEs $h_1$, $h_2$, and $h_3$, we have

1. According to (1) in Theorem 1, it follows that

$$h_1 \circ h_2 \circ h_3 = \bigcup_{\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}, \gamma_1 \neq 1, \gamma_2 \neq 1, \gamma_3 \neq 0} \left\{ \frac{\gamma_1 - \gamma_2 + \gamma_2 \gamma_3}{1 - \gamma_3} \right\} \circ h_3$$

2. Thus, $h_1 \circ h_2 \circ h_1 = h_1 \circ h_1 \circ h_2$. If $\gamma_1 \leq \gamma_2 \gamma_3, \gamma_2 \neq 0, \gamma_3 \neq 0$.
Let be further verified in terms of these two operations. Equations hold only under the given precondition.

Proof. For two HFEs $A_{env}(1)$, $A_{env}(2)$, we have

$$h_1 \odot h_2 \odot h_3 = \bigcup_{y_1, y_2, y_3 \in \psi} \left\{ \frac{y_1 - y_2}{y_3} \right\} \qquad \left( y_1 \geq y_2, y_2 \neq 1 \right)$$

$$h_1 \odot (h_2 \odot h_3) = h_1 \odot \bigcup_{y_1, y_2, y_3 \in \psi} \left\{ \frac{y_1}{y_2} \right\} \qquad \left( y_1 \leq y_2, y_2 \neq 0 \right)$$

This completes the proof.

It should be noted that in the above theorems, the equations hold only under the given precondition. Moreover, the relationship between IFVs and HFEs can be further verified in terms of these two operations.

**Theorem 6.** Let $h_1$ and $h_2$ be two HFEs, then

1. $A_{env}(h_1 \odot h_2) = A_{ax}(h_1) \odot A_{ax}(h_2)$;
2. $A_{env}(h_1 \odot h_2) = A_{ax}(h_1) \odot A_{ax}(h_2)$.

Proof. For two HFEs $h_1$ and $h_2$, we have

1. $A_{env}(h_1 \odot h_2)$
   $$= \bigcup_{y_1, y_2 \in \psi} \left\{ \frac{y_1 - y_2}{1 - y_2} \right\} \qquad \left( y_1 \geq y_2, y_2 \neq 1 \right)$$

2. $A_{env}(h_1 \odot h_2)$
   $$= \bigcup_{y_1, y_2 \in \psi} \left\{ \frac{y_1}{1 - y_2} \right\} \qquad \left( y_1 \leq y_2, y_2 \neq 0 \right)$$

Thus the proof is completed.

Theorem 6 further reveals that the subtraction and division operations defined for HFSs are consistent with the ones for HFEs.

**Theorem 7.** For two HFEs $h_1$ and $h_2$, the followings are valid:

1. $h_1' \odot h_2' = (h_1 \odot h_2)'$;
2. $h_1' \odot h_2' = (h_1 \odot h_2)'$.

Proof. For two HFEs $h_1$ and $h_2$, we have

1. $h_1' \odot h_2' = \bigcup_{y_1, y_2, y_3 \in \psi} \left\{ \frac{1 - y_1}{1 - y_2} \right\} \qquad \left( 1 - y_1 \neq 0, 1 - y_2 \neq 0 \right)$

2. $h_1' \odot h_2' = \bigcup_{y_1, y_2, y_3 \in \psi} \left\{ \frac{1 - y_1}{1 - y_2} \right\} \qquad \left( 1 - y_1 \neq 0, 1 - y_2 \neq 0 \right)$

which complete the proof of the theorem.

**Example 1.** Consider two HFEs $h_1 = (0.3, 0.2)$ and $h_2 = (0.1, 0.2)$. According to the first equation in Definition 7, we have

- $h_1 \odot h_2 = \left\{ \begin{array}{ll} 0.2 & : 0.1 \\ 0.1 & : 0.2 \\ 0.1 & : 0.2 \\ 0.2 & : 0.1 \\ 0.1 & : 0.1 \\ 0.1 & : 0.1 \end{array} \right\}$
- $h_1 \odot h_2 = \left\{ \begin{array}{ll} 0.1 & : 0.1 \\ 0.2 & : 0.0 \end{array} \right\}$

In addition, $h_1' = (0.7, 0.8)$, and $h_2' = (0.9, 0.8)$. Then,
via the second equation in Definition 7, we obtain
\[ h'_1 \circ h'_2 = \left\{ \frac{0.2}{0.3}, \frac{0.1}{0.3}, \frac{0.2}{0.3}, \frac{0.4}{0.3} \right\} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \]
Since \( (h_1 \circ h'_2)^{-1} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\}, \) Therefore, \( (h_1 \circ h'_2)^{-1} = (h'_1 \circ h'_2) \), which verifies the second equation of Theorem 6. The first equation can be verified similarly.

4. Conclusion

In this paper, we have introduced the subtraction operation and division operation over HFSs. Several operational laws of these two operations have been given. The relationships between IFSs and HFSs have been further verified in terms of these two operations. In addition, the relationships between these two operations have also been established. The operations can be immediately extended into interval-valued hesitant fuzzy sets and dual hesitant fuzzy sets. The subtraction and division operations are significantly important in forming the integral theoretical framework of HFS and may have many practical applications in decision making.

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