On the characterizing properties of the permanental polynomials of graphs
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ABSTRACT
Let G be a graph and A(G) the adjacency matrix of G. The permanental polynomial of G is defined as \( \pi(G, x) = \Delta \text{per}(xI - A(G)) \). If two graphs G and H have the same permanental polynomial, then G is called a per-cospectral mate of H. A graph G is said to be characterized by its permanental polynomial if all the per-cospectral mates of G are isomorphic to G. It is shown that complete graphs, stars, regular complete bipartite graphs, and odd cycles are characterized by their permanental polynomials. We prove that in general the permanental polynomial cannot characterize the paths and even cycles. In particular, for each \( l \geq 1 \) and \( m \geq 2 \), we can find non-isomorphic per-cospectral mates of \( P_{4l+3} \) and \( C_{4m} \), respectively. When we restrict our consideration to connected graphs, both the paths and even cycles \( C_{4l+2} \) are characterized by their permanental polynomials.

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1. Introduction

The permanent of an \( n \times n \) matrix M with entries \( m_{ij} \) (\( i, j = 1, 2, \ldots, n \)) is defined by

\[
\text{per}(M) = \sum_{\sigma} \prod_{i=1}^{n} m_{\sigma(i)},
\]

where the sum is taken over all permutations \( \sigma \) of \( \{1, 2, \ldots, n\} \). In strong contrast to determinants, computing permanents, even of matrices in which all entries are 0 or 1, is \#P-complete [16]. Permanent plays an important role in combinatorics. For example, the permanent of a \((0,1)\)-matrix enumerates perfect matchings in bipartite graphs [13].

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Let $G$ be a graph on $n$ vertices and $A(G)$ the adjacency matrix of $G$. The characteristic polynomial of $G$ is defined by

$$
\phi(G, x) = \det(xI - A(G)),
$$

which can be expressed in the coefficient form

$$
\phi(G, x) = \sum_{k=0}^{n} a_k(G)x^{n-k}.
$$

In analogy to Eq. (2), one defines the permanent polynomial of $G$, $\pi(G, x)$, as the permanent of the characteristic matrix of $A(G)$, i.e.,

$$
\pi(G, x) = \text{per}(xI - A(G)).
$$

In what follows, in parallel to Eq. (3), we write the permanent polynomial in the coefficient form

$$
\pi(G, x) = \sum_{k=0}^{n} b_k(G)x^{n-k}.
$$

The roots of the permanent polynomial of $G$ are called the permanent roots of $G$. Recall that the spectrum $S(G)$ of $G$ is defined as the multiset of characteristic roots of $G$. Analogously, Borowiecki [2] defined the per-spectrum $pS(G)$ of $G$ as the multiset of permanent roots of $G$.

It seems that the permanent polynomials of graphs were first considered by Turner [15]. He in fact considered a graph polynomial which generalizes both the permanent and characteristic polynomials. The permanent polynomials of graphs were first systematically studied by Merri et al. [14], and the study of analogous objects in chemical literature were started by Kasum et al. [12].

Borowiecki [2] studied the permanent roots of graphs, and showed that $G$ has $pS(G) = \{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\}$ if and only if $G$ is a bipartite graph without cycles of length $4k$ ($k = 1, 2,\ldots$), where $i^2 = -1$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of $G$. In [3], Borowiecki and Jóźwiak posed a problem: characterize those graphs whose permanent roots are pure imaginary or zero. Yan and Zhang [17] gave a partial solution to this problem. They obtained that if $G$ is a bipartite graph containing no subgraphs which are even subdivisions of $K_{2,3}$, then the permanent roots of $G$ are pure imaginary or zero.

Gutman and Cash [10] and Chen [6] obtained some relations between the coefficients of the permanent and characteristic polynomials of some chemical graphs, such as benzenoid hydrocarbons, fullerenes, 4–6 fullerenes, toroidal fullerenes, and coronoid hydrocarbons. Cash developed a computer-aided method for the calculation of the permanent polynomials of molecular graphs, and applied it to a variety of benzenoid hydrocarbons [4] and fullerenes [5]. Recently, Belardo et al. [1] gave some formulas which express the permanent polynomial of any square matrix in terms of the permanent polynomial of weighted digraphs.

If two graphs $G$ and $H$ have the same permanent polynomial, they are called per-cospectral [3], and $H$ is called a per-cospectral mate of $G$. Note that the permanent polynomial is preserved under permutation similarity:

$$
\text{per}(xI - P^{-1}A(G))P = \text{per}(P^{-1}(xI - A(G))P) = \text{per}(xI - A(G))
$$

for all $n$ by $n$ permutation matrix $P$. It follows that if two graphs $G_1$ and $G_2$ are isomorphic, then they are per-cospectral. A natural question will be asked: Does $G_1$ and $G_2$ having the same permanent polynomial always imply that $G_1$ and $G_2$ are isomorphic? The answer is negative. We found that the smallest example of connected non-isomorphic per-cospectral graphs is the pair $(H_1, H_2)$ shown in Fig. 1, where $\pi(H_1, x) = \pi(H_2, x) = x^6 + 7x^4 - 2x^3 + 12x^2 - 4x + 4$. The smallest such disconnected pair is $(K_{1,3} \cup K_2, P_5 \cup K_1)$, where $\pi(K_{1,3} \cup K_2, x) = \pi(P_3 \cup K_1, x) = x^6 + 4x^4 + 3x^2$. 
For any graph polynomial, it is of interest to determine its ability to characterize graphs. A graph $G$ is said to be characterized (or determined) by its permanental polynomial if any graph having the same permanental polynomial as $G$ is isomorphic to $G$. For graphs on less than six vertices, no pair with the same permanental polynomial exists, so each of these graphs is characterized by its permanental polynomial.

In [14], Merris et al. formulated that the permanental polynomial seems a little better than the characteristic polynomial when it comes to distinguishing graphs which are not trees, since the permanental polynomial distinguishes the five pairs of cospectral graphs (i.e. graphs with the same spectrum) of [11]. Motivated by Merris et al.’s statement, we intend to investigate whether the permanental polynomial really performs better than the characteristic polynomial when we use them to distinguish graphs.

Recall that a graph is said to be characterized (or determined) by its spectrum if any graph having the same spectrum as $G$ is isomorphic to $G$. Clearly, a graph $G$ is characterized by its spectrum is equivalent to $G$ is characterized by its characteristic polynomial. We know that the characteristic polynomial cannot characterize stars [7]. But we will see that stars can be characterized by the permanental polynomial. We find that graphs characterized by the characteristic polynomial are not necessarily characterized by the permanental polynomial. It was shown that the characteristic polynomial characterizes the paths and cycles [7]. However, we prove that in general the paths and even cycles cannot be characterized by the permanent polynomial.

We define three classes of graphs: (a) $\mathcal{U}_0(n, q)$ (see Fig. 2(a)); (b) $\mathcal{U}_1(n, q)$ (see Fig. 2(b)); and (c) $\mathcal{U}_2(n, n_1, n_2, n_3)$ (see Fig. 2(c)), where $0 \leq q \leq \left\lfloor \frac{n-4}{2} \right\rfloor$, $n_1$ is a non-negative integer, and both $n_2$ and $n_3$ are positive integers. All of them are unicyclic on $n$ vertices with girth 4.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and characterizing properties of the permanental polynomials of graphs, and show that complete graphs, stars, regular complete bipartite graphs and odd cycles are characterized by their permanental polynomials. In Section 3, we prove that the paths are characterized by their permanental polynomials when we restrict our attention to connected graphs. However, the statement is no longer true if “connected” is deleted from the hypothesis. In particular, we show that $P_{4l+3}$ and $P_l \cup \mathcal{U}_0(3l + 3, l - 1)$ are per-cospectral (here $G \cup H$ denotes the disjoint union of two graphs $G$ and $H$). In Section 4, we give some characterizing properties of even cycles, and show that even cycles $C_{4l+2}$ are characterized by their permanental polynomials when we restrict our consideration to connected graphs. Moreover, we prove that for each $l \geq 2$, $C_{4l}$ and $\mathcal{U}_0(4l, 2l - 2)$ are per-cospectral. In the final section, we give a remark on the permanental polynomial of $C_{4l+2}$.
2. Some graphs characterized by their permanental polynomials

By the definition of permanent, we immediately obtain the following result.

**Theorem 2.1.** Let $G$ be a graph with components $G_1, G_2, \ldots, G_\omega$. Then $\pi(G, x) = \prod_{i=1}^{\omega} \pi(G_i, x)$.

A subgraph $H$ of a graph $G$ is said to be a Sachs subgraph if each component of $H$ is a single edge or a cycle. Merris et al. [14] obtained a modified Sachs theorem on the permanental polynomial of a graph.

**Theorem 2.2.** Let $G$ be a graph with $\pi(G, x) = \sum_{k=0}^{n} b_k(G) x^{n-k}$. Then

$$b_k(G) = (-1)^k \sum_{H} 2^{c(H)}, \quad 1 \leq k \leq n,$$

where the sum is taken over all Sachs subgraphs $H$ of $G$ on $k$ vertices, and $c(H)$ is the number of cycles in $H$.

By Theorem 2.2, we have that if $G$ is bipartite, then $b_{2k}(G) \geq 0$ and $b_{2k+1}(G) = 0$ for all $k \geq 0$. In fact, Borowiecki and Jóźwiak [3] have obtained the following.

**Theorem 2.3.** $G$ is bipartite if and only if $b_i(G) = 0$ for all odd $i$.

From now on, for a bipartite graph $G$ on $n$ vertices, we may write $\pi(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{2k}(G) x^{n-2k}$.

**Theorem 2.4** [2]. Let $G$ be a graph with spectrum $S(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then the per-spectrum $pS(G)$ of $G$ is $\{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\}$ if and only if $G$ is a bipartite graph without cycles of length $4k$ ($k = 1, 2, \ldots$), where $i^2 = -1$.

Note that all the characteristic roots of a graph are real. By Theorem 2.4, if $G$ is a bipartite graph without cycles of length $4k$ ($k = 1, 2, \ldots$), then all the permanental roots of $G$ are pure imaginary or 0.

An $r$-matching in a graph $G$ is a set of $r$ edges, no two of which have a vertex in common. The number of $r$-matchings in $G$ will be denoted by $p(G, r)$. By convention we assume that $p(G, 0) = 1$.

**Lemma 2.5** [9]. Let $G$ be a graph with an edge $e = uv$. Then

$$p(G, r) = p(G \setminus e, r) + p(G \setminus \{u, v\}, r - 1).$$

**Lemma 2.6** [8]. Let $G$ be a graph with $n$ vertices and $m$ edges, and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. Then

(i) $p(G, 1) = m$.
(ii) $p(G, 2) = \binom{m}{2} - \sum_{i=1}^{n} \binom{d_i}{2}$.
(iii) $p(G, 3) = \binom{m}{3} - (m-2) \sum_{i=1}^{n} \binom{d_i}{2} + 2 \sum_{i=1}^{n} \binom{d_i}{3} + \sum_{ij \in E(G)} (d_i - 1)(d_j - 1) - t$, where $t$ is the number of triangles of $G$.

Obviously, the Sachs subgraph on three vertices is precisely a triangle and the Sachs subgraphs on four vertices are of two kinds: two isolated edges and a quadrangle. By Theorem 2.2 and Lemma 2.6 (ii), we obtain the following result.
**Lemma 2.7.** Let $G$ be a graph with $n$ vertices and $m$ edges, and let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. Then

\[
\begin{align*}
  b_0(G) &= 1, \\
  b_1(G) &= 0, \\
  b_2(G) &= m, \\
  b_3(G) &= -2t, \\
  b_4(G) &= \frac{m^2}{2} - \sum_{i=1}^{n} \left( \frac{d_i}{2} \right) + 2q,
\end{align*}
\]

where $t$ and $q$ denote respectively the numbers of triangles and quadrangles in $G$.

The following characterizing properties of the permanental polynomials of graphs play an important role in proving which graphs are determined by the permanental polynomial.

**Lemma 2.8.** The following can be deduced from the permanental polynomial of a graph $G$:

(i) The number of vertices.
(ii) The number of edges.
(iii) The number of triangles.
(iv) The length of the shortest odd cycle.
(v) The number of the shortest odd cycles.
(vi) Whether $G$ is bipartite.

**Proof.** By Lemma 2.7, items (i), (ii) and (iii) can be easily proved. By Theorem 2.2, it is not difficult to see that the length of the shortest odd cycle is equal to the minimum odd number $k$ such that $b_k(G) \neq 0$, and the number of the shortest odd cycles is equal to $-\frac{b_k(G)}{2}$, where $k$ is the length of the shortest odd cycle. Thus (iv) and (v) are proved. Item (vi) is obtained immediately from Theorem 2.3. □

Now we are in position to show that complete graphs, stars, regular complete bipartite graphs, and odd cycles are characterized by their permanental polynomials, respectively.

**Proposition 2.9.** The complete graphs $K_n$ are characterized by their permanental polynomials.

**Proof.** Suppose that a graph $G$ has the same permanental polynomial as $K_n$. By Lemma 2.8 (i) and (ii), $G$ has $n$ vertices and $\binom{n}{2}$ edges, which implies that $G$ must be $K_n$. □

**Proposition 2.10.** The stars $S_n$ are characterized by their permanental polynomials.

**Proof.** Since $S_n$ contains no Sachs subgraph on more that two vertices, we have $\pi(S_n, x) = x^n + (n - 1)x^{n-2}$. Suppose that a graph $G$ has the same permanental polynomial as $S_n$. Then, by Lemma 2.8 (i) and (ii), $G$ has $n$ vertices and $n - 1$ edges. Since $b_3(G) = b_4(G) = \cdots = b_n(G) = 0$, $G$ has no Sachs subgraphs with $\geq 3$ vertices by Theorem 2.2 and contains no cycles. Thus $G$ is a tree. It is obvious that any two edges of $G$ are adjacent, for otherwise $G$ contains one Sachs subgraph on 4 vertices and $b_4(G) > 0$, a contradiction. Since $G$ is not a triangle, $G$ is isomorphic to $S_n$. □

**Proposition 2.11.** The regular complete bipartite graphs $K_{p,p}$ are characterized by their permanental polynomials.

**Proof.** Suppose that $G$ has the same permanental polynomial as $K_{p,p}$. By Lemma 2.8 (i), (ii) and (vi), $G$ is a bipartite graph with $2p$ vertices and $p^2$ edges. Since $K_{p,p}$ has a perfect matching, we have $b_{2p}(G) = b_{2p}(K_{p,p}) > 0$. By Theorem 2.2, $G$ has a Sachs subgraph on $2p$ vertices. Since a Sachs subgraph of $G$ on $2p$ vertices is a spanning subgraph of $G$ whose components are even cycles or single edges, $G$ contains a perfect matching. Therefore, $G$ has a bipartition with two parts of the same size. Since $G$ has $p^2$ edges, $G$ must be isomorphic to $K_{p,p}$. □
Proposition 2.12. The odd cycles $C_{2l+1}$ are characterized by their permanental polynomials.

Proof. It is easy to see that $C_3$ is characterized by its permanental polynomial. So we assume that $l \geq 2$. Suppose that a graph $G$ has the same permanental polynomial as $C_{2l+1}$. By Lemma 2.8 (i) and (ii), $G$ has $2l + 1$ vertices and $2l + 1$ edges. Clearly, $b_3(C_{2l+1}) = b_5(C_{2l+1}) = \cdots = b_{2l-1}(C_{2l+1}) = 0$ and $b_{2l+1}(C_{2l+1}) = -2$. Since $\pi(G, x) = \pi(C_{2l+1}, x)$, we have $b_3(G) = b_5(G) = \cdots = b_{2l-1}(G) = 0$ and $b_{2l+1}(G) = -2$. Thus $G$ contains no cycle of length $2k + 1$ for $1 \leq k < l$. It follows that the Sachs subgraphs of order $2l + 1$ in $G$ are precisely cycles of length $2l + 1$. Since $b_{2l+1}(G) = -2$, $G$ has exactly one cycle of length $2l + 1$. Note that $G$ has $2l + 1$ vertices and $2l + 1$ edges. Therefore, $G$ is isomorphic to $C_{2l+1}$. □

3. Paths

Clearly, the Sachs subgraphs on $2k$ vertices of a path are precisely $k$-matchings. We have known $p(P_n, k) = \binom{n-k}{k}$ (see, for example, [9]). By Theorem 2.2, we immediately obtain the permanental polynomial of $P_n$.

Theorem 3.1. Let $P_n$ be a path on $n$ vertices. Then

$$\pi(P_n, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k}.$$ (8)

Firstly, we show that $P_n$ can be characterized by its permanental polynomial when we restrict graphs considered to be connected.

Theorem 3.2. Let $G$ be a connected graph. If $G$ has the same permanental polynomial as $P_n$, then $G$ is isomorphic to $P_n$.

Proof. Suppose that $\pi(G, x) = \pi(P_n, x)$. By Lemma 2.8 (i) and (ii), $G$ has $n$ vertices and $n-1$ edges, and $G$ is a tree since $G$ is connected. Assume that the spectrum of $P_n$ is $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then, by Theorem 2.4, the per-spectrum of $P_n$ is $\{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\}$, where $i^2 = -1$. Since $\pi(G, x) = \pi(P_n, x)$, $G$ and $P_n$ have the same per-spectrum, i.e. $pS(G) = \{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\}$. Using Theorem 2.4 again, we see that the spectrum of $G$ is $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Thus $G$ and $P_n$ are cospectral. Since $P_n$ is determined by its spectrum [7], $G$ must be isomorphic to $P_n$. □

If “connected” is deleted from the hypothesis, Theorem 3.2 is no longer true. In particular, we show that for each $l \geq 1$, $P_{4l+3}$ cannot be characterized by its permanental polynomial. That is, we can find a non-isomorphic per-cospectral mate of $P_{4l+3}$. Before proving this result, we make the following lemma. From now on, $G \cup H$ denotes the disjoint union of two graphs $G$ and $H$.

Lemma 3.3. $2p(P_l \cup P_{l+1} \cup P_{2l+2}, k) = p(P_l \cup P_{l+1} \cup P_{2l+3}, k) + p(P_{l+1} \cup P_{l+1} \cup P_{2l+1}, k)$ for $l \geq 0$ and $k \geq 0$.

Proof. We use induction on $(l, k)$. It is easy to see that $2p(P_l \cup P_{l+1} \cup P_{2l+2}, 0) = p(P_l \cup P_{l+1} \cup P_{2l+3}, 0) + p(P_{l+1} \cup P_{l+1} \cup P_{2l+1}, 0) = 2$ and $2p(P_l \cup P_{l+1} \cup P_{2l+2}, 1) = p(P_l \cup P_{l+1} \cup P_{2l+3}, 1) + p(P_{l+1} \cup P_{l+1} \cup P_{2l+1}, 1) = 8l$.

Suppose that the lemma is true for $(l', k') < (l, k)$. For $k \geq 2$, by repeated application of Lemma 2.5, we give three expressions as follows:

$$2p(P_l \cup P_{l+1} \cup P_{2l+2}, k) = p(P_l \cup P_{l+1} \cup P_{2l+2}, k) + p(P_l \cup P_{l+1} \cup P_{2l+2}, k)$$
$$= p(P_l \cup P_{l+1} \cup P_{2l+1}, k) + p(P_l \cup P_{l+1} \cup P_{2l+2}, k - 1)$$
$$+ p(P_l \cup P_l \cup P_{2l+2}, k) + p(P_{l-1} \cup P_{2l+2}, k - 1)$$
and

\[
p(P_{l+1} \cup P_{l+1} \cup P_{2l+1}, k) = p(P_{l+1} \cup P_{l} \cup P_{2l+1}, k) + p(P_{l+1} \cup P_{l-1} \cup P_{2l+1}, k - 1)
+ p(P_{l} \cup P_{l+1} \cup P_{2l+1}, k - 1) + p(P_{l-1} \cup P_{l-1} \cup P_{2l+1}, k - 2).
\]

By simple calculations, we obtain

\[
2p(P_{l} \cup P_{l+1} \cup P_{2l+2}, k) - p(P_{l} \cup P_{l} \cup P_{2l+3}, k) - p(P_{l+1} \cup P_{l+1} \cup P_{2l+1}, k)
= 2p(P_{l-1} \cup P_{l} \cup P_{2l}, k - 2) - p(P_{l} \cup P_{l} \cup P_{2l-1}, k - 2) - p(P_{l-1} \cup P_{l-1} \cup P_{2l+1}, k - 2).
\]

By the induction hypothesis, we have 2

\[
2p(P_{l} \cup P_{l+1} \cup P_{2l+2}, k) - p(P_{l} \cup P_{l} \cup P_{2l+3}, k) - p(P_{l+1} \cup P_{l+1} \cup P_{2l+1}, k)
= 2p(P_{l-1} \cup P_{l} \cup P_{2l}, k - 2) - p(P_{l} \cup P_{l} \cup P_{2l-1}, k - 2) - p(P_{l-1} \cup P_{l-1} \cup P_{2l+1}, k - 2).
\]

By simple calculations, we obtain

\[
\pi(P_{4l+3}, x) = \pi(P_{l} \cup \%0(3l + 3, l - 1), x).
\]

\textbf{Theorem 3.4.} For every \(l \geq 1\), we have \(\pi(P_{4l+3}, x) = \pi(P_{l} \cup \%0(3l + 3, l - 1), x)\).

\textbf{Proof.} By Theorems 2.2 and 3.1, we have \(\pi(P_{7}, x) = \pi(P_{l} \cup \%0(6, 0), x) = x(8^6 + 6x^4 + 10x^2 + 4).\) So we assume that \(l \geq 2\). Clearly, \(b_0(P_{4l+3}) = b_0(P_{l} \cup \%0(3l + 3, l - 1)) = 1\) and \(b_2(P_{4l+3}) = b_2(P_{l} \cup \%0(3l + 3, l - 1)) = 4l + 2\). By Theorem 2.3, we only need to show that \(b_{2k}(P_{4l+3}) = b_{2k}(P_{l} \cup \%0(3l + 3, l - 1))\) for \(2 \leq k \leq 2l + 1\).

Let \(C\) be the unique 4-cycle of \(\%0(3l + 3, l - 1)\). Let \(u\) be a vertex of \(C\) with degree 2 in \(\%0(3l + 3, l - 1)\), and \(u\) and \(w\) the neighbors of \(v\) (see Fig. 3). To compute \(b_{2k}(P_{l} \cup \%0(3l + 3, l - 1))\), we classify the Sachs subgraphs \(H\) on \(2k\) vertices of \(P_{l} \cup \%0(3l + 3, l - 1)\) into three kinds:

(a) \(H\) contains both edges incident to \(v\). Then \(H\) is the union of \(C\) and a \((k - 2)\)-matching of \(P_{l} \cup \%0(3l + 3, l - 1)\)\(V(C)\).

(b) \(H\) contains exactly one edge incident to \(v\). Then \(H\) is the union of \(uv\) and a \((k - 1)\)-matching of \(P_{l} \cup \%0(3l + 3, l - 1)\)\(\{u, v\}\), or the union of \(vw\) and a \((k - 1)\)-matching of \(P_{l} \cup \%0(3l + 3, l - 1)\)\(\{v, w\}\).

(c) \(H\) contains no edge incident to \(v\). Then \(H\) is a \(k\)-matching of \(P_{l} \cup \%0(3l + 3, l - 1)\)\(\{v\}\).

Hence, by Theorem 2.2, we have

\[
b_{2k}(P_{l} \cup \%0(3l + 3, l - 1)) = 2p(P_{l} \cup P_{l-1} \cup P_{2l}, k - 2) + p(P_{l} \cup P_{l-1} \cup P_{2l+2}, k - 1)
+ p(P_{l-1} \cup P_{l+1} \cup P_{2l}, k - 1) + p(P_{l-1} \cup P_{3l+2}, k).
\]
By repeated application of Lemma 2.5, we have

\[
b_{2k}(P_{4l+3}) = p(P_{4l+3}, k) \\
= p(P_{l+1} \cup P_{3l+2}, k) + p(P_l \cup P_{3l+1}, k - 1) \\
= p(P_l \cup P_{3l+2}, k) + p(P_{l-1} \cup P_{3l+2}, k - 1) + p(P_l \cup P_{3l+1}, k - 1) \\
= p(P_l \cup P_{3l+2}, k) + p(P_{l-1} \cup P_l \cup P_{2l+2}, k - 1) + p(P_{l-1} \cup P_{l-1} \cup P_{2l+1}, k - 2) \\
+ p(P_l \cup P_{l+1} \cup P_{2l}, k - 1) + p(P_l \cup P_{l} \cup P_{2l-1}, k - 2).
\]

By simple calculations, we have

\[
b_{2k}(P_l \cup \gamma_0(3l + 3, l - 1)) - b_{2k}(P_{4l+3}) = 2p(P_l \cup P_{l-1} \cup P_{2l}, k - 2) \\
- p(P_{l-1} \cup P_{l-1} \cup P_{2l+1}, k - 2) \\
- p(P_l \cup P_l \cup P_{2l-1}, k - 2).
\]

By Lemma 3.3, we obtain \(b_{2k}(P_l \cup \gamma_0(3l + 3, l - 1)) = b_{2k}(P_{4l+3})\) for \(2 \leq k \leq 2l + 1\). □

For other cases of paths \(P_n\), i.e. \(n \not\equiv 3 \pmod{4}\), the situation seems to be complicated. We use Maple to test paths on less than 20 vertices. The result is that \(P_n\) (\(n \leq 6\)), \(P_9\), \(P_{10}\), \(P_{12}\), \(P_{16}\) and \(P_{18}\) can be characterized by the permanental polynomial, while \(P_8\), \(P_{13}\), \(P_{14}\) and \(P_{17}\) cannot be characterized. Non-isomorphic per-cospectral mates of \(P_8\), \(P_{13}\), \(P_{14}\) and \(P_{17}\) are given in Fig. 4, respectively.

In fact, \(\pi(P_8, x) = \pi(G_1, x) = (x^6 + 6x^4 + 9x^2 + 1)(x^2 + 1)\), \(\pi(P_{13}, x) = \pi(G_2, x) = (x^7 + 7x^5 + 14x^3 + 7x)(x^5 + 5x^4 + 6x^2 + 1)\), \(\pi(P_{14}, x) = \pi(G_3, x) = (x^8 + 9x^6 + 26x^4 + 24x^2 + 1)(x^4 + 3x^2 + 1)(x^2 + 1)\), and \(\pi(P_{17}, x) = \pi(G_4, x) = x(x^8 + 9x^6 + 27x^4 + 30x^2 + 9)(x^8 + 7x^6 + 15x^4 + 10x^2 + 1)\).

4. Even cycles

In this section, we first show that if \(G\) is a connected graph with the same permanental polynomial as \(C_{4l+2}\), then \(G\) is isomorphic to \(C_{4l+2}\). However, if “connected” is deleted from the hypothesis, the statement is no longer true and some examples are given. Next, we find a non-isomorphic per-cospectral mate of \(C_{4l}\) for each \(l \geq 2\).

It is easy to see that the Sachs subgraphs on \(2k\) (\(2k < n\)) vertices of \(C_n\) are \(k\)-matchings. We have known \(p(C_n, k) = \frac{n}{n-k} \binom{n-k}{k}\) (see, for example, [9]). By Theorem 2.2, we immediately obtain the permanental polynomial of \(C_n\).
Theorem 4.1. Let $C_n$ be a cycle on $n$ vertices. Then
\[
\pi(C_n, x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} + b_n(C_n),
\]  \hspace{1cm} (9)
where $b_n(C_n) = -2$ if $n$ is odd, and 4 otherwise.

When we restrict our consideration to connected graphs, even cycles $C_{4l+2}$ are characterized by their permanental polynomials. Before proving this result, we need the following lemmas.

Lemma 4.2. Suppose that $H$ is a regular graph of degree $r$ with $n$ vertices and $m$ edges, and $G$ has the same permanental polynomial as $H$. If $G$ and $H$ have the same number of quadrangles, then $G$ is also regular of degree $r$.

Proof. Since $G$ has the same permanental polynomial as $H$, $G$ and $H$ have the same number of vertices and edges, and $b_i(H) = b_i(G)$ for $i = 0, 1, \ldots, n$.

By Lemma 2.7, we have
\[
\frac{1}{2} \sum_{i=1}^{n} d_i = \frac{1}{2} \sum_{i=1}^{n} r,
\]  \hspace{1cm} (10)
\[
\binom{m}{2} - \sum_{i=1}^{n} \left( \binom{d_i}{2} + 2q \right) = \binom{m}{2} - \sum_{i=1}^{n} \binom{r}{2} + 2q,
\]  \hspace{1cm} (11)
where $(d_1, d_2, \ldots, d_n)$ is the degree sequence of $G$, and $q$ is the number of quadrangles in $G$. Starting from Eq. (11), and applying Eq. (10) in the last step, we have
\[
0 = \sum_{i=1}^{n} \left[ \binom{d_i}{2} - \binom{r}{2} \right]
= \frac{1}{2} \sum_{i=1}^{n} \left( d_i^2 - d_i - r^2 + r \right)
= \frac{1}{2} \sum_{i=1}^{n} \left( (d_i^2 - 2rd_i + r^2) - (d_i - r + 2rd_i - 2r^2) \right)
= \frac{1}{2} \sum_{i=1}^{n} \left[ (d_i - r)^2 + (2r - 1)(d_i - r) \right]
= \frac{1}{2} \sum_{i=1}^{n} (d_i - r)^2 + \frac{1}{2} (2r - 1) \sum_{i=1}^{n} (d_i - r)
= \frac{1}{2} \sum_{i=1}^{n} (d_i - r)^2.
\]
This implies that $d_i = r$ for $i = 1, 2, \ldots, n$. That is to say, $G$ is regular of degree $r$. \hfill \Box

Lemma 4.3. Let $G$ be a graph containing no quadrangle. If $G$ has the same permanental polynomial as $C_{2n}$ ($n \geq 3$), then $G$ is isomorphic to $C_{2n}$.

Proof. By Lemma 4.2, $G$ is regular of degree 2. It follows from Lemma 2.8 (vi) that $G$ is a bipartite graph. Suppose, to the contrary, that $G$ is not isomorphic to $C_{2n}$. Then $G$ is the disjoint union of even cycles, i.e., $G = C_{2k_1} \cup C_{2k_2} \cup \cdots \cup C_{2k_s}$, where $s \geq 2$, $\sum_{i=1}^{s} 2k_i = 2n$, and $k_i > 2$ for $1 \leq i \leq s$. 
Clearly, \( G \) itself is a spanning Sachs subgraph of \( G \), which contributes \( 2^s \) to \( b_{2n}(G) \). On the other hand, \( G \) has \( 2^s \) perfect matchings. By Theorem 2.2, these perfect matchings contribute \( 2^s \) to \( b_{2n}(G) \). Hence \( b_{2n}(G) \geq 2^s + 2^s \geq 8 \), which contradicts the fact that \( b_{2n}(G) = b_{2n}(C_{2n}) = 4 \). \( \square \)

From Lemma 4.3, we immediately obtain the following result.

**Corollary 4.4.** If \( G \) has the same permanental polynomial as \( C_{2n} \), and \( G \) is not isomorphic to \( C_{2n} \) (\( n \geq 3 \)), then \( G \) must contain a quadrangle.

**Lemma 4.5.** Suppose that \( G \) is a connected graph with the same permanental polynomial as \( C_{2n} \) (\( n \geq 3 \)), and \( G \) is not isomorphic to \( C_{2n} \), then \( G \) is unicyclic with girth 4, and \( G \) has exactly two perfect matchings \( M_1 \) and \( M_2 \) such that the symmetric difference \( M_1 \Delta M_2 = C_4 \).

**Proof.** By Lemma 2.8 (i), (ii) and (vi), \( G \) is a bipartite graph with \( 2n \) vertices and \( 2n \) edges. Clearly, \( G \) is unicyclic since \( G \) is connected. It follows from Corollary 4.4 that \( G \) contains a quadrangle. Hence the girth of \( G \) is 4. The fact \( b_{2n}(G) = b_{2n}(C_{2n}) = 4 \) implies that \( G \) has exactly two perfect matchings. In fact, if \( G \) has exactly one perfect matching, then, by Theorem 2.2, we have \( b_{2n}(G) = 1 \), a contradiction. If \( G \) has three perfect matchings \( M_1, M_2 \) and \( M_3 \), then the symmetric difference \( M_1 \Delta M_2 \) contains at least one even cycle, say \( C \). Then \( C \cup (M_1 \setminus V(C)) \) is a spanning Sachs subgraph of \( G \), which contributes 2 to \( b_{2n}(G) \). Since each perfect matching of \( M_1, M_2 \) and \( M_3 \) contributes 1 to \( b_{2n}(G) \), we have \( b_{2n}(G) \geq 2 + 1 + 1 + 1 = 5 \), a contradiction. This completes the proof. \( \square \)

**Lemma 4.6.** Let \( G \) be a connected graph with the same permanental polynomial as \( C_{2n} \) (\( n \geq 3 \)). If \( G \) is not isomorphic to \( C_{2n} \), then the degree sequence of \( G \) is \((1^2, 3^2, 2^{2n-4})\), where \( a^k \) means \( a, a, \ldots, a \) \( (k \) times).\)

**Proof.** By Lemma 2.8 (i), (ii) and (vi), \( G \) is a bipartite graph with \( 2n \) vertices and \( 2n \) edges. It follows from Lemma 4.5 that \( G \) is unicyclic with girth 4. Since \( \pi(C_{2n}, x) = \pi(G, x) \), we have \( b_{2k}(C_{2n}) = b_{2k}(G) \) for \( k = 0, 1, \ldots, n \). Suppose that the degree sequence of \( G \) is \((2 + t_1, 2 + t_2, \ldots, 2 + t_{2n})\), where \( t_1, t_2, \ldots, t_{2n} \) are integers. Since \( G \) has \( 2n \) edges, we obtain \( \sum_{i=1}^{2n} (2 + t_i) = 4n \), i.e., \( \sum_{i=1}^{2n} t_i = 0 \). By Lemma 2.7, we have \( b_4(C_{2n}) = \left(\frac{2n}{2}\right) - 2n \), and

\[
b_4(G) = \left(\frac{2n}{2}\right) - \sum_{i=1}^{2n} \left(\frac{2 + t_i}{2}\right) + 2
= \left(\frac{2n}{2}\right) - \frac{1}{2} \sum_{i=1}^{2n} \left(t_i^2 + 3t_i + 2\right) + 2
= \left(\frac{2n}{2}\right) - \frac{1}{2} \sum_{i=1}^{2n} t_i^2 - \frac{3}{2} \sum_{i=1}^{2n} t_i - \frac{1}{2} (2 \cdot 2n) + 2
= \left(\frac{2n}{2}\right) - \frac{1}{2} \sum_{i=1}^{2n} t_i^2 - 2n + 2.
\]

Since \( b_4(C_{2n}) = b_4(G) \), we have \( \sum_{i=1}^{2n} t_i^2 = 4 \). The possible values for \((t_1, t_2, \ldots, t_{2n})\) are (i) \((1^4, 0^{2n-4})\); (ii) \((-1)^4, 0^{2n-4})\); (iii) \((1, (-1)^3, 0^{2n-4})\); (iv) \((1^2, (-1)^2, 0^{2n-4})\); (v) \((1^3, -1, 0^{2n-4})\); (vi) \((2, 0^{2n-1})\); and (vii) \((-2, 0^{2n-1})\). If (i) occurs, then the degree sequence of \( G \) is \((3^2, 2^{2n-4})\). We see at once that the degree sum of \( G \) is \( 3 \times 4 + 2(2n - 4) = 4n + 4 \neq 4n \), a contradiction. By a similar argument, we show that none of (ii), (iii), (v) and (vi) can occur. If (vii) occurs, then \( G \) contains an isolated vertex, contradicting the connectivity of \( G \). Hence \((t_1, t_2, \ldots, t_{2n})=\left(1^2, (-1)^2, 0^{2n-4}\right)\). It follows that the degree sequence of \( G \) is \((1^2, 3^2, 2^{2n-4})\). \( \square \)
Now we are in position to present one of the main results of this section.

**Theorem 4.7.** Let $G$ be a connected graph with the same permanental polynomial as $C_{4l+2}$ ($l \geq 1$). Then $G$ is isomorphic to $C_{4l+2}$.

**Proof.** By Lemmas 4.5 and 4.6, it is not difficult to verify that if $\pi(G,x) = \pi(C_6,x)$, then $G$ is isomorphic to $C_6$. So we assume that $l \geq 2$. Suppose, to the contrary, that $G$ is not isomorphic to $C_{4l+2}$. Then, by Lemma 4.6, the degree sequence of $G$ is $(1^2, 2^2, 2^{4l-2})$. It follows from Lemma 4.5 that $G$ must be one of the following graphs: (a) $\mathcal{V}_0(4l+2, 2q)$, $1 \leq q \leq l-1$; (b) $\mathcal{V}_n(4l+2, 2q)$, $1 \leq q \leq l-1$; and (c) $\mathcal{V}_n'(4l+2, n_1, n_2, n_3)$, where $\mathcal{V}_n'(4l+2, n_1, n_2, n_3)$ are those graphs $\mathcal{V}_n(4l+2, n_1, n_2, n_3)$ with the property that $\mathcal{V}_n(4l+2, n_1, n_2, n_3) \setminus \{C_4\}$ has exactly one perfect matching (here $C_4$ is the unique cycle of $\mathcal{V}_n(4l+2, n_1, n_2, n_3)$). We will show that none of the above three classes of graphs has the same permanental polynomial as $C_{4l+2}$. The proof falls naturally into three parts.

**Claim 1.** $C_{4l+2}$ cannot have the same permanental polynomial as $\mathcal{V}_0(4l+2, 2q)$ for $1 \leq q \leq l-1$.

**Proof.** Since $p(C_n,k) = \frac{n}{n-k} \binom{n-k}{k}$, we have $b_{d_1}(C_{4l+2}) = p(C_{4l+2}, 2l) = 4l^2 + 4l + 1$. By a similar argument as computing $b_{2k}(P_l \cup \mathcal{V}_0(3l+3, l-1))$ in the proof of Theorem 3.4, we classify the Sachs subgraphs on $2k$ vertices of $\mathcal{V}_0(4l+2, 2q)$ into three kinds, and we obtain

$$b_{2k}(\mathcal{V}_0(4l+2, 2q)) = 2p(P_{2q} \cup P_{4l-2q-2}, k-2) + p(P_{2q} \cup P_{4l-2q}, k-1) + p(P_{4l+1}, k).$$

(12)

Therefore, by replacing $k$ with $2l$ in Eq. (12) and repeated application of $p(P_n,k) = \binom{n-k}{k}$, we have

$$b_{d_1}(\mathcal{V}_0(4l+2, 2q)) = 2p(P_{2q} \cup P_{4l-2q-2}, 2l-2) + p(P_{2q} \cup P_{4l-2q}, 2l-1) + p(P_{4l+1}, 2l)$$

$$= 2 \left[ p(P_{2q}, q)p(P_{4l-2q-2}, 2l-q-2) + p(P_{2q}, q-1)p(P_{4l-2q-2}, 2l-q-1) + p(P_{2q}, q-1)p(P_{4l-2q}, 2l-q) + p(P_{2q}, q)p(P_{4l-2q}, 2l-q-1) + p(P_{4l+1}, 2l)$$

$$= 2 \left[ \binom{4l-2q-2 - (2l-q-2)}{2l-q-2} + \binom{2q - (q-1)}{q-1} \right] + \binom{4l-2q - (2l-q-1)}{2l-q-1}$$

$$+ \binom{2q - (q-1)}{q-1} + \binom{4l-2q - (2l-q-2)}{2l-q-2} + \binom{2q + 2-q}{q} + \binom{4l+1 - 2l}{2l}$$

$$= 4q^2 + (4 - 8l)q + 8l^2 + 2.$$

Hence $b_{d_1}(\mathcal{V}_0(4l+2, 2q)) - b_{d_1}(C_{4l+2}) = 4q^2 + (4 - 8l)q + 4l^2 - 4l + 1$. If $l = 2$, then $q = 1$, and $b_8(\mathcal{V}_0(10, 2)) - b_8(\mathcal{V}_0(10)) \neq 1$. For $l \geq 3$, it is not difficult to verify that the quadratic function $f(q) = 4q^2 + (4 - 8l)q + 4l^2 - 4l + 1$ has no root in the interval $[1, l-1]$. In fact, the axis of symmetry is at $q = l - \frac{1}{2}$, and $f(l-1) = 1 > 0$. It follows that $b_{d_1}(\mathcal{V}_0(4l+2, 2q)) \neq b_{d_1}(C_{4l+2})$. Therefore, $\pi(C_{4l+2}, x) \neq \pi(\mathcal{V}_0(4l+2, 2q), x)$ for $1 \leq q \leq l-1$. □

**Claim 2.** $C_{4l+2}$ cannot have the same permanental polynomial as $\mathcal{V}_1(4l+2, 2q)$ for $1 \leq q \leq l-1$.

**Proof.** It follows from $p(C_n,k) = \binom{n-k}{k}$ that $b_{d_6}(C_{4l+2}) = p(C_{4l+2}, 3) = \frac{33l^3 - 24l^2 - 8l + 6}{3}$. Clearly, the Sachs subgraphs on 6 vertices in a bipartite graph are of three kinds: one is the disjoint union of a
It follows that $\pi(C_{4l+2}, x) \neq \pi(C_{4l+2}, x)$. Therefore, $\pi(C_{4l+2}, x) \neq \pi(C_{4l+2}, x)$ for $1 \leq q \leq l - 1$. □

**Claim 3.** $C_{4l+2}$ cannot have the same permanental polynomial as $\mathcal{W}_2'(4l + 2, n_1, n_2, n_3)$.

**Proof.** It is not difficult to verify that $4l + 5 \leq \sum_{i=1}^{n_3} d_i (d_i - 1) \leq 4l + 7$. Moreover, the left-hand side equality holds if and only if $n_1 \geq 2, n_2 \geq 2$ and $n_3 = 1$, and the right-hand side equality holds if and only if $n_1 = 0, n_2 \geq 2$ and $n_3 \geq 2$.

By Lemma 2.6 (iii), we have

$$\frac{32l^3 - 24l^2 - 32l + 27}{3} \leq p(\mathcal{W}_2'(4l + 2, n_1, n_2, n_3), 3) \leq \frac{32l^3 - 24l^2 - 32l + 33}{3}.$$ 

Therefore, by Theorem 2.2, we have

$$b_6(\mathcal{W}_2'(4l + 2, n_1, n_2, n_3)) = p(\mathcal{W}_2'(4l + 2, n_1, n_2, n_3), 3) + 2(4l + 2 - 5) \geq \frac{32l^3 - 24l^2 - 32l + 27}{3} + 8l - 6 \geq \frac{32l^3 - 24l^2 - 8l + 9}{3} > b_6(C_{4l+2}).$$

It follows that $\pi(C_{4l+2}, x) \neq \pi(C_{4l+2}, x)$. □

From Claims 1, 2 and 3, the theorem is proved. □

Theorem 4.7 is no longer true if “connected” is deleted from the hypothesis. Non-isomorphic per-cospectral mates of $C_{14}, C_{18}$ and $C_{30}$ are given in Fig. 5, respectively. In fact, $\pi(C_{14}, x) = \pi(G_5, x) = (x^8 + 9x^6 + 26x^4 + 25x^2 + 4)(x^6 + 5x^4 + 6x^2 + 1), \pi(C_{18}, x) = \pi(G_6, x) = (x^6 + 6x^4 + 9x^2 + 1)^2(x^6 + 6x^4 + 9x^2 + 4), \pi(C_{30}, x) = \pi(G_7, x) = (x^6 + 9x^6 + 26x^4 + 24x^2 + 1)^2(x^6 + 3x^2 + 1)^2(x^6 + 6x^4 + 9x^2 + 4).$$
In the following, we show that the permanental polynomial cannot characterize even cycles $C_{4l}$ for each $l \geq 2$. In fact, we can find a non-isomorphic per-cospectral mate of $C_{4l}$ for $l \geq 2$. Before proving this result, we give the following lemma.

**Lemma 4.8.** $p(P_{2l} \cup P_{2l+2}, k) = p(P_{2l+1} \cup P_{2l+1}, k)$ for $l \geq 0$ and $0 \leq k \leq 2l$.

**Proof.** We proceed by induction on $(l, k)$ for $l \geq 0$ and $0 \leq k \leq 2l$. It is easy to see that $p(P_{2l} \cup P_{2l+2}, 0) = p(P_{2l+1} \cup P_{2l+1}, 0) = 1$ and $p(P_{2l} \cup P_{2l+2}, 1) = p(P_{2l+1} \cup P_{2l+1}, 1) = 4l$.

Suppose that the lemma is true for $(l', k') < (l, k)$, where $k' \leq 2l'$. For $2 \leq k \leq 2l$, by repeated application of Lemma 2.5, we give two expressions as follows:

\[
p(p_{2l} \cup p_{2l+2}, k) = p(p_{2l} \cup p_{2l+1}, k) + p(p_{2l} \cup p_{2l}, k-1)
\]
\[
= p(p_{2l} \cup p_{2l+1}, k) + p(p_{2l-1} \cup p_{2l}, k-1) + p(p_{2l-2} \cup p_{2l}, k-2)
\]

and

\[
p(p_{2l+1} \cup p_{2l+1}, k) = p(p_{2l} \cup p_{2l+1}, k) + p(p_{2l-1} \cup p_{2l+1}, k-1)
\]
\[
= p(p_{2l} \cup p_{2l+1}, k) + p(p_{2l-1} \cup p_{2l}, k-1) + p(p_{2l-1} \cup p_{2l-1}, k-2).
\]

By simple calculations, we have

\[
p(p_{2l} \cup p_{2l+2}, k) - p(p_{2l+1} \cup p_{2l+1}, k) = p(p_{2l-2} \cup p_{2l}, k-2) - p(p_{2l-1} \cup p_{2l-1}, k-2).
\]

By the induction hypothesis, we have $p(p_{2l-2} \cup p_{2l}, k-2) = p(p_{2l-1} \cup p_{2l-1}, k-2)$ for $2 \leq k \leq 2l$. It follows that $p(P_{2l} \cup P_{2l+2}, k) = p(P_{2l+1} \cup P_{2l+1}, k)$ for $2 \leq k \leq 2l$. By the principle of induction, we have proved the lemma. □

**Theorem 4.9.** For every $l \geq 2$, we have $\pi(C_{4l}, x) = \pi(\%0(4l, 2l - 2), x)$.

**Proof.** Obviously, $b_0(C_{4l}) = b_0(\%0(4l, 2l - 2)) = 1, b_2(C_{4l}) = b_2(\%0(4l, 2l - 2)) = 4l$, and $b_4l(C_{4l}) = b_4l(\%0(4l, 2l - 2)) = 4$. By Theorem 2.3, we only need to show that $b_{2k}(C_{4l}) = b_{2k}(\%0(4l, 2l - 2))$ for $2 \leq k \leq 2l - 1$.

By a similar argument as computing $b_{2k}(P_l \cup \%0(3l + 3, l - 1))$ in the proof of Theorem 3.4, we classify the Sachs subgraphs on $2k$ vertices of $\%0(4l, 2l - 2)$ into three kinds, and we obtain

\[
b_{2k}(\%0(4l, 2l - 2)) = 2p(P_{2l-2} \cup P_{2l-2}, k-2) + 2p(P_{2l-2} \cup P_{2l}, k-1) + p(P_{4l-1}, k).
\]

By repeated application of Lemma 2.5, we have

\[
b_{2k}(C_{4l}) = p(C_{4l}, k)
\]
\[
= p(P_{4l}, k) + p(P_{4l-2}, k-1)
\]
\[
= p(P_{4l-1}, k) + p(P_{4l-2}, k-1) + p(P_{4l-2}, k-1)
\]
\[
= p(P_{4l-1}, k) + 2p(P_{4l-2}, k-1)
\]
\[
= p(P_{4l-1}, k) + 2(p(P_{2l-1} \cup P_{2l-1}, k-1) + p(P_{2l-2} \cup P_{2l-2}, k-2)).
\]

It follows from Lemma 4.8 that $b_{2k}(C_{4l}) = b_{2k}(\%0(4l, 2l - 2))$ for $2 \leq k \leq 2l - 1$. This completes the proof. □

By a similar argument as the proof of Theorem 4.7, we can show that if $G$ is a connected graph with the same permanental polynomial as $C_{4l}$ ($l \geq 2$) and $G$ is not isomorphic to $C_{4l}$, then $G$ must be isomorphic to $\%0(4l, 2l - 2)$.

From Theorem 4.9, we find a connected non-isomorphic per-cospectral mate of $C_{4l}$ for each $l \geq 2$. Note that $C_{4l}$ may have a disconnected non-isomorphic per-cospectral mate.
Example 4.10. \( C_{12} \) has the same permanental polynomial as \( G_8 \) (see Fig. 6). In fact, \( \pi (C_{12}, x) = \pi (G_8, x) = (x^3 + 8x^6 + 18x^4 + 8x^2 + 1)(x^4 + 4x^2 + 4) \).

5. A remark

An interesting phenomenon is that \( C_{4l+2} \) and \( \emptyset (4l + 2, 2l - 2) \) have very similar permanental polynomials. More specifically, we have the following.

Theorem 5.1. For \( l \geq 1 \), we have \( \pi (C_{4l+2}, x) = \pi (\emptyset (4l + 2, 2l - 2), x) - x^2 \).

Before proving Theorem 5.1, we make the following lemma.

Lemma 5.2. \( p(P_{2l+1} \cup P_{2l+3}, k) = p(P_{2l} \cup P_{2l+4}, k) \) for \( l \geq 0 \) and \( 0 \leq k \leq 2l \); and \( p(P_{2l-1} \cup P_{2l+1}, 2l - 1) = p(P_{2l-2} \cup P_{2l+2}, 2l - 1) - 1 \) for \( l \geq 1 \).

Proof. We use induction on \((l, k)\) to prove the first part of the lemma. It is easy to see that \( p(P_{2l+1} \cup P_{2l+3}, 0) = p(P_{2l} \cup P_{2l+4}, 0) = 1 \) and \( p(P_{2l+1} \cup P_{2l+3}, 1) = p(P_{2l} \cup P_{2l+4}, 1) = 4l + 2 \).

Suppose that the first part of the lemma is true for \((l', k') < (l, k)\), where \( k' \leq 2l' \). For \( 2 \leq k \leq 2l \), by repeated application of Lemma 2.5, we give two expressions as follows.

\[
p(P_{2l+1} \cup P_{2l+3}, k) = p(P_{2l+1} \cup P_{2l+2}, k) + p(P_{2l+1} \cup P_{2l+1}, k - 1)
= p(P_{2l+1} \cup P_{2l+1}, k) + p(P_{2l+1} \cup P_{2l}, k - 1)
+ p(P_{2l} \cup P_{2l+1}, k - 1) + p(P_{2l+1} \cup P_{2l+1}, k - 2)
= p(P_{2l} \cup P_{2l+1}, k) + p(P_{2l-1} \cup P_{2l+1}, k - 1) + p(P_{2l+1} \cup P_{2l}, k - 1)
+ p(P_{2l} \cup P_{2l}, k - 1) + p(P_{2l-1} \cup P_{2l}, k - 2) + p(P_{2l-1} \cup P_{2l+1}, k - 2)
\]

and

\[
p(P_{2l} \cup P_{2l+4}, k) = p(P_{2l} \cup P_{2l+3}, k) + p(P_{2l} \cup P_{2l+2}, k - 1)
= p(P_{2l} \cup P_{2l+2}, k) + p(P_{2l} \cup P_{2l+1}, k - 1)
+ p(P_{2l-1} \cup P_{2l+2}, k - 1) + p(P_{2l-2} \cup P_{2l+2}, k - 2)
= p(P_{2l} \cup P_{2l+1}, k) + p(P_{2l} \cup P_{2l}, k - 1) + p(P_{2l} \cup P_{2l+1}, k - 1)
+ p(P_{2l-1} \cup P_{2l}, k - 1) + p(P_{2l-1} \cup P_{2l}, k - 2) + p(P_{2l-2} \cup P_{2l+2}, k - 2).
\]

By simple calculations, we have

\[
p(P_{2l+1} \cup P_{2l+3}, k) - p(P_{2l} \cup P_{2l+4}, k) = p(P_{2l-1} \cup P_{2l+1}, k - 2) - p(P_{2l-2} \cup P_{2l+2}, k - 2).
\]

By the induction hypothesis, we have \( p(P_{2l-1} \cup P_{2l+1}, k - 2) = p(P_{2l-2} \cup P_{2l+2}, k - 2) \) for \( 2 \leq k \leq 2l \). It yields that \( p(P_{2l+1} \cup P_{2l+3}, k) = p(P_{2l} \cup P_{2l+4}, k) \) for \( 2 \leq k \leq 2l \). By the principle of induction, we have proved the first part of the lemma.
By \( p(P_q, k) = \binom{n-k}{k} \), we obtain \( p(P_{2l-1} \cup P_{2l+1}, 2l - 1) = p(P_{2l-1}, l - 1) p(P_{2l+1}, l) = l^2 + l \) and \( p(P_{2l-2} \cup P_{2l+2}, 2l - 1) = p(P_{2l-2}, l - 1) p(P_{2l+2}, l) + p(P_{2l-2}, l - 2) p(P_{2l+2}, l + 1) = l^2 + 1. \) This completes the proof. \( \square \)

**Proof (of Theorem 5.1).** It is easy to verify that \( C_6 \) satisfies Theorem 5.1. In fact, \( \pi(C_6, x) = x^6 + 6x^4 + 9x^2 + 4 \) and \( \pi(\%_0(6, 0), x) = x^6 + 6x^4 + 10x^2 + 4. \) So we assume that \( l \geq 2. \) Clearly, \( b_0(C_{4l+2}) = b_0(\%_0(4l + 2, 2l - 2)) = 1, b_2(C_{4l+2}) = b_2(\%_0(4l + 2, 2l - 2)) = 4l + 2, \) and \( b_{4l+2}(C_{4l+2}) = b_{4l+2}(\%_0(4l+2, 2l-2)) = 4. \) By Theorem 2.3, we only need to show that \( b_{2k}(C_{4l+2}) = b_{2k}(\%_0(4l+2, 2l-2)) \) for \( 2 \leq k \leq 2l - 1 \) and \( b_{4l}(C_{4l+2}) = b_{4l}(\%_0(4l+2, 2l-2)) - 1. \)

Replacing \( q \) by \( l - 1 \) in Eq. (12), we have

\[
b_{2k}(\%_0(4l+2, 2l-2)) = 2p(P_{2l-2} \cup P_{2l}, k - 2) + p(P_{2l-2} \cup P_{2l+2}, k - 1) + p(P_{2l} \cup P_{2l-1}, k - 1) + p(P_{4l+1}, k)
\]

By repeated application of Lemma 2.5, we have

\[
b_{2k}(C_{4l+2}) = p(C_{4l+2}, k) = p(P_{4l+2}, k) + p(P_{4l}, k - 1)
\]

By simple calculations, we have

\[
b_{2k}(C_{4l+2}) - b_{2k}(\%_0(4l+2, 2l-2)) = p(P_{2l-1} \cup P_{2l-1}, k - 2) - p(P_{2l-2} \cup P_{2l}, k - 2) + p(P_{2l-1} \cup P_{2l}, k - 1) - p(P_{2l-2} \cup P_{2l+2}, k - 1).
\]

It follows from Lemmas 4.8 and 5.2 that \( b_{2k}(C_{4l+2}) - b_{2k}(\%_0(4l+2, 2l-2)) = 0 \) for \( 2 \leq k \leq 2l - 1 \) and \( b_{4l}(C_{4l+2}) - b_{4l}(\%_0(4l+2, 2l-2)) = -1. \) \( \square \)

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**References**


